A short mathematical description of Keynes's general theory of unemployment, prices and interest

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Second version, with illustrated example

1 Setup

1.1 Income

Let y be the income - or total production of the economy, measured in some real production unit. It is assumed that, at a given state of society, the income is uniquely determined by the number of persons employed, n:

$$y = y(n),$$

where the function y is monotonically increasing. A certain part of the income is consumed,

$$c = c(y),$$

where the function c is monotonically increasing, but its growth is slow enough so that the surplus of income over consumption,

$$s(y) = y - c(y)$$

is increasing in y as well.

The expected profitability of investment is described by the marginal efficiency of capital:

$$e = e(v, q).$$

This is the expected rate of return on the least profitable investment when the income of the investment sector is v. It is assumed that investments are undertaken in order of decreasing efficiency, so that e is decreasing in v. The other argument determining the expected profit is the level of optimism of investors, q, which is defined so that e is increasing in q.

1.2 Wages and prices

Production cost, in money units, per unit of output - i.e., money wages and the cost of other production factors - increases with employment:

$$p = p(n).$$

This is a result of the increasing bargaining power of labor as well as the non-homogeneousness of labor and other production factors. As income increases various production bottlenecks are encountered and as a result the cost of the production unit increases. At full employment, $n = n_{\mathrm{F}}$, the function approaches a vertical asymptote.

Since income is an increasing function of employment, wages can also be expressed as an increasing function of income:

$$p = p(y).$$

1.3 Monetary

The money in the economy, m, is used either for conducting transactions or for storing wealth. The quantities used for those purposes are m_{T} and m_{S}. Thus:

$$m = m_{\mathrm{T}} + m_{\mathrm{S}}.$$

The amount of money used for transactions depends linearly on income and on wages:

$$m_{\mathrm{T}} = Ap(y)y,$$

while the amount of money used for storing wealth depends on the interest rate and on production unit cost:

$$m_{\mathrm{S}} = m_{\mathrm{S}}(r, p).$$

It is decreasing in r and increasing in p.

2 Analysis

The optimism of the investors, q, and the total amount of money, m, are assumed exogenous.

Equilibrium is the point at which

$$e(v, q) = r.$$

Thus, the equation determining the equilibrium state is:

$$m = a\,p(y)\,y + m_{\mathrm{S}}\Big(e\big(s(y), q\big), p(y)\Big),$$

with a single unknown, y. The right hand side is monotonically increasing in y since both summands are increasing in y.

Seen as a function of the exogenous parameters, the solution, $y(m, q)$, is increasing in both.

2.1 Employment-inflation tradeoff

If profitable investments exist at full employment, i.e.,

$$e\Big(s\left(y\left(n_{\mathrm{F}}\right)\right), q\Big) > 0,$$

then y increases to $y(n_{\mathrm{F}})$ as m increases. As unemployment decreases to zero, $p(y)$ increases asymptotically. In such a situation, any increase in m increases employment and prices. As the economy approaches full employment, the main effect of an increase in the money supply is an increases in prices.

2.2 The zero interest rate lower bound

If, on the other hand, $e(s(y), q) = 0$ for some $y_{\mathrm{Z}} < y\left(n_{\mathrm{F}}\right)$, then for large enough m the interest rate approaches zero. At this point only marginally profitable investments remain and as a result no increase in y beyond y_{Z} occurs as m increases. This is the so-called "zero lower bound" situation. At this point any additional increase in the money supply is absorbed as wealth storage. Therefore the wealth storage function m_{S} must have an asymptote at $r = 0$.

3 Example

3.1 Specification

Let y range between 0 (no production) and 1 (maximum production). Let

$$\begin{aligned} c(y) &= y_0 + (1-\alpha)(y - y_0), \\ e(v, q) &= e_0\left(q - \frac{v}{v_0}\right), \\ p(y) &= \frac{p_0}{(1-y)^d}, \end{aligned}$$

and

$$m_{\mathrm{S}}(r, p) = \frac{B}{r}\frac{p}{p_0}.$$

3.2 Derivation

For this consumption function,

$$s(y) = y - c(y) = \alpha(y - y_0).$$

Substituting the surplus function into the expected profitability of investment,

$$e\big(s(y), q\big) = e(q)\left(1 - \frac{y}{y_{\mathrm{Z}}(q)}\right),$$

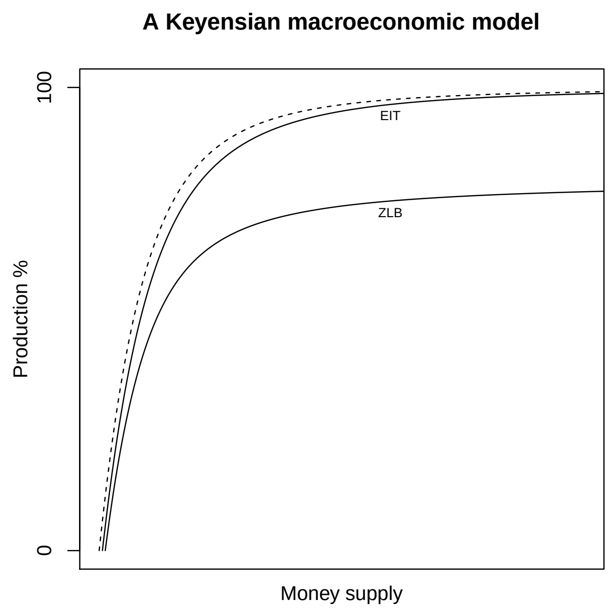

Figure 1: Money supply vs. production in a Keynesian model. The solid lines show the functional relationships for the case of an inflation-unemployment tradeoff and for the case of the zero lower bound. The dashed line shows the relationship that would hold if all money was always used for transactions, regardless of the interest rate. The parameter values used when drawing the graphs are $y_0 = 0.5, \alpha = 0.1, d = 0.5, p_0 = 1, A = 1, B = 1, v_0 = 0.01$, and $q = 3$ for the zero lower bound curve (lower line), and $q = 10$ for the inflation-unemployment tradeoff curve (upper solid line).

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where

$$e(q) = e_0 \left(q + \alpha \frac{y_0}{v_0} \right),$$

and

$$y_Z = \frac{e(q)}{e_0} \frac{v_0}{\alpha}.$$

The value of y_Z determines whether the economy can reach full employment. If $y_Z > 1$ then the expected rate of return on investment at full employment is positive and the unemployment can be pushed to arbitrarily low values by increasing the money supply. This is the situation shown by the top solid line in Figure 1.

If, on the other hand, $y_Z < 1$, then the interest rate drops to zero as the product approaches y_Z and increasing the money supply cannot push production beyond this level (the lower solid line in Figure 1).

The level of optimism, q, determines which of the two situations occurs: high values of q,

$$q > \alpha \frac{1 - y_0}{v_0},$$

are associated with the first situation and low values of q are associated with the second situation.